Reconstruction of a Critically Sampled Cosine Function

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1 Useful Definitions

Since there are several ways to define Fourier and inverse Fourier transforms, the forms used in this paper are listed here. Additionally the definitions of the sinc function along with its Fourier transform, the rect function are listed for clarity. Finally a common expression for the Dirac delta function is also included.

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-i\omega t}dt \tag{1}$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{i\omega t}d\omega \tag{2}$$

$$\operatorname{sinc}(t) = \frac{\sin(\pi t)}{\pi t} \tag{3}$$

$$\operatorname{rect}(\omega) = \begin{cases} 1 & -\pi < \omega < \pi \\ \frac{1}{2} & |\omega| = \pi \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

$$\delta(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-jkt}\,dt \tag{5}$$

2 Proof

The basic problem stems from the Nyquist sampling theorem where if a signal is sampled more than twice as fast as its bandwidth, then the signal may be perfectly reconstructed. A problem occurs when a signal is critically sampled so that the bandwidth is exactly one half of the sampling rate. A simple example occurs when a sinusoid is sampled this way. It turns out that at the critical sampling rate, the measured amplitude of the sinusoid will be dependent upon the relative phase between the sinusoid and its sampling clock. Certainly one can see that if a sinusoid is sampled on its zero crossings, that the samples will all be identically zero, yielding no information. If the same sinusoid is sampled at its extrema, then the sequence $\cdots, -1, 1, -1, 1, -1, 1, \cdots$ will be found, assuming the sinusoid's amplitude is one. The problem addressed here is to show that when the alternating ones sequence is used to reconstruct a continuous signal that the proper sinusoid is in fact reconstituted.

For the derivation, the sample period is taken as one, so the sample frequency is 2π To start, the temporal function is written in terms of sinc functions.

$$x(t) = \sum_{n=-\infty}^{\infty} (-1)^n \operatorname{sinc}(t-n) \tag{6}$$

Now move over to the frequency domain via Fourier transformation.

$$X(\omega) = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} (-1)^n \operatorname{sinc}(t-n) e^{-j\omega t}\, dt \tag{7}$$

And swap the order of integration and summation.

$$X(\omega) = \sum_{n=-\infty}^{\infty} (-1)^n \int_{-\infty}^{\infty} \operatorname{sinc}(t-n) e^{-j\omega t}\, dt \tag{8}$$

Make a change of variable by substituting $t = \hat{t} + n$.

$$X(\omega) = \sum_{n=-\infty}^{\infty} (-1)^n \int_{-\infty}^{\infty} \operatorname{sinc}(\hat{t}) e^{-j\omega(\hat{t}+n)}\, d\hat{t} \tag{9}$$

Factor out the non $\hat{t}$ dependent portion from the integral.

$$X(\omega) = \sum_{n=-\infty}^{\infty} (-1)^n\, e^{-j\omega n} \int_{-\infty}^{\infty} \operatorname{sinc}(\hat{t}) e^{-j\omega \hat{t}}\, d\hat{t} \tag{10}$$

Recognizing the integral as just the Fourier transform of sinc($\hat{t}$).

$$X(\omega) = \sum_{n=-\infty}^{\infty} (-1)^n e^{-j\omega n} \operatorname{rect}(\omega) \tag{11}$$

Substituting $e^{j\pi}$ for -1 and factoring out $\operatorname{rect}(\omega)$.

$$X(\omega) = \operatorname{rect}(\omega) \sum_{n=-\infty}^{\infty} e^{jn\pi} e^{-j\omega n} \tag{12}$$

Combine exponentials.

$$X(\omega) = \operatorname{rect}(\omega) \sum_{n=-\infty}^{\infty} e^{jn(\pi-\omega)} \tag{13}$$

To resolve the summation, use the Poisson Summation formula. It is:

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{2\pi jnt} f(t)\, dt \tag{14}$$

Substituting $exp(jn(\pi - \omega))$ for $f(n)$.

$$\sum_{n=-\infty}^{\infty} e^{jn(\pi-\omega)} = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{2\pi jnt} e^{jt(\pi-\omega)}\, dt \tag{15}$$

Combine exponentials.

$$\sum_{n=-\infty}^{\infty} e^{jn(\pi-\omega)} = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-j(\omega-(2n+1)\pi))t}\, dt \tag{16}$$

Compare this integral with the definition of the Dirac delta function and equate $\omega-(2n+1)\pi$ with k in the definition.

$$\int_{-\infty}^{\infty} e^{-j(\omega-(2n+1)\pi))t}\, dt = 2\pi\, \delta\left(\omega - (2n+1)\pi\right) \tag{17}$$

The summation is

$$\sum_{n=-\infty}^{\infty} e^{jn(\pi-\omega)} = \sum_{n=-\infty}^{\infty} 2\pi\, \delta\left(\omega - (2n+1)\pi\right) \tag{18}$$

Thus, the summation is seen to produce a series of Dirac delta functions occurring at odd multiples of π. Substitute the new summation formula into $X(\omega)$.

$$X(\omega) = \operatorname{rect}(\omega) \sum_{n=-\infty}^{\infty} 2\pi\, \delta(\omega - (2n+1)\pi) \tag{19}$$

Since only two of the delta functions coincide with the nonzero portion of the $\mathrm{rect}(\omega)$ function, the infinite sum only needs two terms.

$$X(\omega) = 2\pi\,\mathrm{rect}(\omega)\left[\delta(\omega - \pi) + \delta(\omega + \pi)\right] \tag{20}$$

Convert back to the time domain via the inverse Fourier transform.

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\,\mathrm{rect}(\omega)\left[\delta(\omega - \pi) + \delta(\omega + \pi)\right] e^{j\omega t}\,d\omega \tag{21}$$

Perform the integration. I.e., the delta functions serve to sample the rest of the integrand.

$$\begin{aligned} x(t) &= \frac{1}{2}\left(e^{j\pi t} + e^{-j\pi t}\right) && (22)\\ &= \cos(\pi t) && (23) \end{aligned}$$

Thus, the resulting cosine function has a frequency of π, which is half of the sampling rate, and it has the correct amplitude.